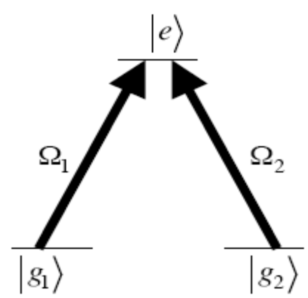
Physics 566 Fall 2010

Problem Set #4
Due: Friday Oct. 8, 2010

Problem 1: Free induction decay (10 Points)
(a) Assume a 2-level atom, initially in the ground state. At t = 0, apply a pulse of light, with oscillation frequency wo, i.e., on resonance. The shape of the pulse is described by an envelope function E(t), with E(0) = E(T) = 0. Assume that $T << \Gamma^{-1}$, so you can ignore spontaneous emission. Find the condition on $\int dt\, E(t)$ from 0 to T such that after the pulse is over, $\rho_{ee} = 1/2$. What is ρ_{eg} (in the rotating frame)?
(b) Take the case of sodium where Γ^{-1} = 16 ns, and the optical wavelength is 589 nm. Take E(t) to be constant from t =0 to t =T, and let T = 100 ps (this is a typical pulse length for a garden variety mode-locked laser). Assume, as we have before, that the atom is a 2-level atom, with the transition dipole moment being along *z*, and that the laser is polarized along *z*. Calculate what intensity (in Watts/cm^2) is required to achieve the π/2 pulse of part (a). Compare this power to the power such that the Rabi frequency $\Omega = \Gamma$.
(c) Starting with the atom as in part (a), after the pulse, calculate the evolution of the density matrix (ρ_{ee} and ρ_{eg}). Describe the evolution. This is the free induction decay due to spontaneous emission.

Problem 2: Dark states (10 points)
Let us consider again a three level "lambda system"



The two ground states are resonantly coupled to the excited state, each with a different Rabi frequency. Taking the two ground states as the zero of energy, then in the RWA (and in the rotating frame) the Hamiltonian is

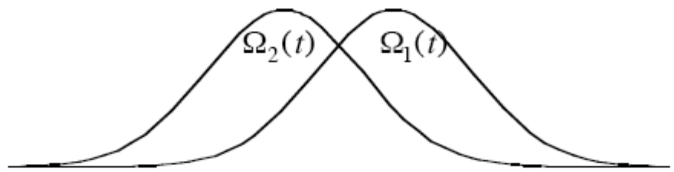
$$\hat{H}_{AL} = -\frac{\hbar}{2}\left[\Omega_1\left(|g_1\rangle\langle e| + |e\rangle\langle g_1|\right) + \Omega_2\left(|g_2\rangle\langle e| + |e\rangle\langle g_2|\right)\right]$$

(a) Find the "dressed states" of this system (i.e. the eigenstates and eigenvalues of the total atom laser system). You should find that one of these states has a **zero** eigenvalue,

$$|Dark\rangle = \Omega_2|g_1\rangle - \Omega_1|g_2\rangle$$

This particular superposition is called a "**dark state**" or uncoupled state because the laser field does not couple it to the excited state. Explain how this can be true.
(b) Adiabatic transfer through the "nonintuitive" pulse sequence. Suppose we want to transfer population from $|g_1\rangle$ to $|g_2\rangle$. Could try to Raman-Rabi flop between these states

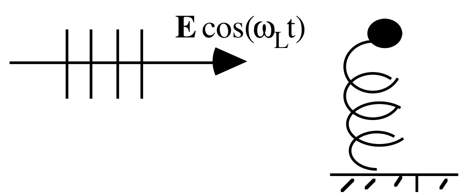
as in the previous problem set. An alternative, and robust method is to use adiabatic passage, *always staying in the local dark state*. This can then be *on resonance*.
Show that if we apply a slowly varying pulse $\Omega_2(t)$ overlapped, but followed by $\Omega_1(t)$ shown below,



we accomplish this transfer. **Hint**: Sketch the dressed state eigenvalues a function of time. Note, the pulse sequence going from $|g_1\rangle \rightarrow |g_2\rangle$ is "counter intuitive" as a realtransition involving absorption and emission. This is quantum mechanics!

Problem 3: Lorentz Classical Model of Absorption and Emission (15 points)

Suppose we were to model an atom as an electron on a spring - i.e. a damped simple harmonic oscillator of mass *m*, with resonance frequency ω_0, and damping constant Γ. Consider driving the oscillator with a monochromatic plane wave, of frequency ω_L.



(a) Show that rate at which the dipole absorbs energy from the field, given by the rate at which the field does work on the charge averaged over one period, is

$$\frac{dW_{abs}}{dt} = \frac{\pi e^2 |\mathbf{E}|^2}{4m} g(\omega_L), \text{ where } g(\omega) = \frac{\Gamma/(2\pi)}{(\omega-\omega_{eg})^2 + \Gamma^2/4} \text{ is the line shape.}$$

Assume near resonance so that $\Delta = \omega_L - \omega_0 << \omega_L, \omega_0$.

(b) The absorption cross section, σ_{abs}, is defined as the rate at which energy is absorbed by an atom, divided by the flux, Φ, of photons incident on the atom, $\Phi \equiv I/\hbar\omega_L$ (i.e. the rate of photons incident on the atom per unit area), where $I = \frac{c}{8\pi}|\mathbf{E}_0|^2$ is the incident intensity (CGS units). Show that the classical model of absorption gives,

$$\sigma_{classical} = \frac{2\pi^2 e^2}{mc} g(\omega_L),$$

Evaluate this on-resonance, for a the parameters associated with Na, where the excitation wavelength is 589 nm and the linewidth (Full width at half-maximum) is 10 MHz.

The ratio of the integrated cross section an atomic transition and that given by the classical model to the quantum mechanical expression with equal resonance frequency and line width is known as the *oscillator* strength of the transition.

(c) From standard texts we have $\sigma_{quantum} = 4\pi^2 \frac{e^2}{\hbar c} \left|\langle e|\mathbf{x}|g\rangle\right|^2 \omega_L\, g(\omega_L)$, where $\langle e|\mathbf{x}|g\rangle$ is the matrix element of the electron position relative to the nucleus for the resonant transition. **Show** that on resonance,

$$f = \frac{\sigma_{quantum}}{\sigma_{classical}} = \frac{2m\omega_0}{\hbar}\left|\langle e|x|g\rangle\right|^2 .$$

Note that $\hbar / 2m\omega_0$ is the square of the characteristic length scale of a quantum simple harmonic oscillator . Thus, the oscillator strength measures the ratio of dipole oscillation amplitude for a two level atom as compared to a simple harmonic oscillator.

Let us now assume that our spring has no *intrinsic* damping associated with it. Consider the scattering of an electromagnetic wave by this oscillating charge. As the charge radiates, the electromagnetic field will carry away energy. This energy must come from the kinetic energy of the accelerating charge. Thus the very act of radiating should "damp" the motion of the charge. This is known as radiation reaction, and will determine a classical decay rate Γ_{class} for the oscillator. In steady state the power radiated by the charge (given by the classical Larmor formula) is equal to the power absorbed.

(d) Assume that the oscillator is damped as Γ_{class}, and show that

$P_{abs} = P_{radiated} \Rightarrow \Gamma_{class} = \frac{2}{3}\frac{e^2}{mc^3}\omega^2 = \frac{2}{3}\, k r_c\, \omega$,where r_c is the classical electron radius.

(e) Show that the quantum mechanical decay rate is related to the classical formula by

$\Gamma_{quantum} = f\,\Gamma_{class}$, where *f* is the oscillator strength.

Problem 4: Radiation reaction and decay of the quantum oscillator (15 points)
Radiation reaction can be shown to be lead to the decay of the quantum mechanical oscillator as well if we work in the *Heisenberg picture*. Start with the total Hamitonian for a two level atom interacting with the quantized field, as discussed in class,

$$\hat{H} = \frac{1}{2}\hbar\omega_{eg}\,\hat{\sigma}_z + \sum_{\mathbf{k}\lambda}\hbar\omega_k \hat{a}^\dagger_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda} - \sum_{\mathbf{k}\lambda}\hbar(g_{\mathbf{k}\lambda}\;\hat{\sigma}_+\hat{a}_{k\lambda} + g^*_{\mathbf{k}\lambda}\;\hat{a}^\dagger_{k\lambda}\hat{\sigma}_-),$$

$$\text{where } g_{\mathbf{k}\lambda} = i\sqrt{\frac{2\pi\hbar\omega_k}{V}}\,\vec{\varepsilon}_{\mathbf{k}\lambda}\cdot\mathbf{d}_{eg}.$$

Note: We have expressed the interaction operator in "normally ordered" form, so that annihilation operators are to the right and creation operators are to the left. We have complete freedom to do this since field and atomic operators commute *at equal times*.

(a) Show that the Heisenberg equations of motion are:

$$\frac{d}{dt}\hat{a}_{k\lambda} = -i\omega_k\hat{a}_{k\lambda} + ig^*_{\mathbf{k}\lambda}\hat{\sigma}_-$$

$$\frac{d}{dt}\hat{\sigma}_- = -i\omega_{eg}\hat{\sigma}_- - i\sum_{\mathbf{k},\lambda} g_{\mathbf{k},\lambda}\hat{\sigma}_z\hat{a}_{\mathbf{k}\lambda}$$

$$\frac{d}{dt}\hat{\sigma}_z = 2i\sum_{k,\lambda}\left(g_{k\lambda}\hat{\sigma}_+\hat{a}_{k\lambda} - g^*_{k\lambda}\hat{a}^\dagger_{k\lambda}\hat{\sigma}_-\right)$$

(b) The first equation is linear in the operators, and so can be formally integrated. Show that

$$\hat{a}_{k\lambda}(t) = \underbrace{\hat{a}_{k\lambda}(0)e^{-i\omega_k t}}_{\hat{a}^{vac}_{k\lambda}(t)} + \underbrace{ig^*_{\mathbf{k}\lambda}\int_0^t dt'\,\hat{\sigma}_-(t')e^{-i\omega_k(t-t')}}_{\hat{a}^{source}_{k\lambda}(t)}$$

is a solution. The first term $\hat{a}^{vac}_{k\lambda}(t)$ is known as the vacuum field operator and $\hat{a}^{source}_{k\lambda}(t)$ is known as the source component due to dipole radiation by the atom.

(c) Show that, in general, given $[\hat{a}_{k\lambda}(t),\hat{a}^\dagger_{k'\lambda'}(t)] = \delta_{\lambda\lambda'}\delta_{kk'}$, unitary evolution implies $\left[\hat{a}_{k\lambda}(0),\hat{a}^\dagger_{k\lambda}(0)\right] = \delta_{\lambda\lambda'}\delta_{kk'}$. Show that the source part alone does not satisfy these relations.

(d) Plug the solution (b) back into the equation for $\hat{\sigma}_z$. Take the Heisenberg state of the system (initial state in the Schrodinger picture) to be $|\Psi\rangle = |\psi\rangle_{atom} \otimes |0\rangle_{field}$, i.e., arbitrary state of the atom plus field in the vacuum. Shown that expectation value satisfies

$$\frac{d}{dt}\langle\hat{\sigma}_z\rangle = -2\sum_{\mathbf{k}\lambda}|g_{\mathbf{k}\lambda}|^2 \int_0^t \langle\hat{\sigma}_+(t)\hat{\sigma}_-(t')\rangle e^{-i\omega_k(t-t')} + c.c$$

(e) Now let us make the Markov approximation. Assume that $\hat{\sigma}_-(t) = \hat{\Sigma}_-(t)e^{-i\omega_{eg}t}$, where $\hat{\Sigma}_-(t)$ is a slowly varying operator on the scale of ω_{eg}. Under this assumption $\langle\hat{\Sigma}_+(t)\hat{\Sigma}_-(t')\rangle \approx \langle\hat{\Sigma}_+(t)\hat{\Sigma}_-(t)\rangle = \langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle$. Use this approximation to show,

$$\frac{d}{dt}\langle\hat{\sigma}_z\rangle = -\Gamma\left(1 + \langle\hat{\sigma}_z\rangle\right),$$

where $\Gamma = \sum_{\mathbf{k}\lambda} 2\pi|g_{\mathbf{k}\lambda}|^2 \delta(\omega_k - \omega_{eg})$ is the Einstein A coefficient!

This is the expected decay! Note that vacuum fluctuations play no role in determining the *rate* of emission. They just initiate the process if the initial dipole moment is zero.

Problem 5: Extra Credit
Momentum and Angular Momentum in the E&M Field (20 points)
From classical electromagnetic field theory we know that conservation laws require that the field carry momentum and angular momentum

$$\mathbf{P} = \int d^3x \left(\frac{\mathbf{E}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})}{4\pi c} \right), \ \mathbf{J} = \int d^3x \left(\mathbf{x} \times \frac{\mathbf{E}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})}{4\pi c} \right).$$

(a) Show that when these quantities become field operators, the momentum operator becomes, $\hat{\mathbf{P}} = \sum_{\mathbf{k},\lambda} \hbar \mathbf{k} \hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda}$; interpret.

(b) Show that $\mathbf{J} = \mathbf{J}_{orb} + \mathbf{J}_{spin}$

where $\mathbf{J}_{orb} = \dfrac{1}{4\pi c} \int d^3x\, E_i(\mathbf{x}) (\mathbf{x} \times \nabla) A_i(\mathbf{x})$, $\mathbf{J}_{spin} = \dfrac{1}{4\pi c} \int d^3x\, (\mathbf{E}(\mathbf{x}) \times \mathbf{A}(\mathbf{x}))$

(c) Show that

$\hat{\mathbf{J}}_{orb} = \sum_{\mathbf{k}\,\mathbf{k}'} \sum_{\lambda} \hat{a}^\dagger_{\mathbf{k}'\lambda} \left(i\hbar \nabla_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{k}') \times \mathbf{k} \right) \hat{a}_{\mathbf{k}\lambda}$, where $\nabla_{\mathbf{k}}$ is the gradient in **k-space**, and

$\hat{\mathbf{J}}_{spin} = \hbar \sum_{\mathbf{k}} \left(\hat{a}^\dagger_{\mathbf{k},+} \hat{a}_{\mathbf{k},+} - \hat{a}^\dagger_{\mathbf{k},-} \hat{a}_{\mathbf{k},-} \right) \mathbf{e}_{\mathbf{k}}$. Interpret these quantities.

(d) The spin of the photon has magnitude S=1, yet there are only two helicity states. Thus we can map the spin angular momentum onto the Bloch(Poincaré) sphere for S=1/2, via

$$\hat{\mathbf{J}}_{spin} = \hat{J}_x \mathbf{e}_x + \hat{J}_y \mathbf{e}_y + \hat{J}_z \mathbf{e}_z,$$

with $J_z = \dfrac{\hbar}{2} (\hat{a}^\dagger_{z+} \hat{a}_{z+} - \hat{a}^\dagger_{z-} \hat{a}_{z-})$, $J_x = \dfrac{\hbar}{2} (\hat{a}^\dagger_{z+} \hat{a}_{z-} + \hat{a}^\dagger_{z-} \hat{a}_{z+})$, $J_y = \dfrac{\hbar}{2i} (\hat{a}^\dagger_{z+} \hat{a}_{z-} - \hat{a}^\dagger_{z-} \hat{a}_{z+})$,

where $(\hat{a}_{z+}, \hat{a}_{z-})$ are the mode operators for positive and negative helicity operators relative to a *space fixed* quantization axis.

(di) Show that these operators satisfy the SU(2) commutation algebra for angular momentum. This relationship is know as the "Schwinger representation" (see Sakauri).

(dii) The mean values of $\hat{J}_x, \hat{J}_y, \hat{J}_z$ are the "Stokes parameters" in classical optics and the Bloch vector components on the Poincaré sphere. Explain the relationship between these operators and the Pauli operators.